

Recoverability and Expectations-Driven Fluctuations*

Ryan Chahrour
Boston College

Kyle Jurado
Duke University

January 15, 2021

Abstract

Time series methods for identifying structural economic disturbances often require disturbances to satisfy technical conditions that can be inconsistent with economic theory. We propose replacing these conditions with a less restrictive condition called recoverability, which only requires that the disturbances can be inferred from the observable variables. As an application, we show how shifting attention to recoverability makes it possible to construct new identifying restrictions for technological and expectational disturbances. In a vector autoregressive example using postwar U.S. data, these restrictions imply that independent disturbances to expectations about future technology are a major driver of business cycles.

JEL classification: E32, D84, C32

Keywords: recoverability, non-fundamentalness, expectations, business cycles

*Chahrour: Department of Economics, Boston College, 140 Commonwealth Avenue, Chestnut Hill, MA 02467 (e-mail: ryan.chahrour@bc.edu); Jurado: Department of Economics, Duke University, 419 Chapel Drive, Durham, NC 27708 (e-mail: kyle.jurado@duke.edu).

1 Introduction

Economists often seek to explain economic fluctuations in terms of exogenous random disturbances to the underlying structure of the economy. A popular empirical strategy, based on the proposal of Sims (1980), is to associate these structural disturbances with linear combinations of contemporaneous forecast errors obtained from an unstructured multivariate time series model, typically a vector autoregression (VAR). Some a priori theoretical restrictions are needed to determine which linear combinations of the forecast errors to choose, but this strategy has the advantage of requiring far fewer restrictions than estimation of a fully specified structural model.

Despite the popularity of this strategy, a number of papers, beginning with Hansen and Sargent (1991), have argued that it is only feasible if the structural disturbances can be expressed as linear combinations of contemporaneous forecast errors — or more precisely, if the disturbances satisfy the technical condition of being *fundamental*. This condition says that the disturbances must be both *causal*, which means that they cannot affect observable variables in advance, and *invertible*, which means that they can be inferred from just current and past (but not future) observables. The reason that requiring fundamentalness poses a problem is that fundamentalness is often inconsistent with economic theory.¹ As a result, practitioners often first check whether disturbances are fundamental in a candidate theoretical model, using tests such as that of Fernández-Villaverde et al. (2007); if not, they then resort to fully structural methods.²

In this paper, we argue that fundamentalness should not be understood as a feasibility condition for identifying structural disturbances using time series methods in the spirit of Sims (1980). The main contribution of this paper is to propose replacing this condition with a less restrictive condition, which only requires that the disturbances can be inferred from the observable variables available to the econometrician. We call this condition *recoverability* and provide a necessary and sufficient condition that is easy to check in any linear candidate model.³ This proposal expands the

¹See Hansen and Sargent (1980), Futia (1981), and Quah (1990) for early examples.

²This is the original remedy proposed by Hansen and Sargent (1991), and has been adopted by a large part of the literature on anticipated disturbances. See the arguments in Schmitt-Grohé and Uribe (2012), Barsky and Sims (2012), and Blanchard et al. (2013).

³We use the term “recoverable” interchangeably with “identified,” though the latter is sometimes reserved for parameters. But like parameter identification (and fundamentalness), recoverability is

set of disturbances that can be identified using these methods, because fundamental disturbances are always recoverable, but not vice versa. The recoverability condition also helps to detect the presence of “omitted variables” problems, because it indicates exactly when the observable variables contain enough information to identify the disturbances of interest.

We are not the first to suggest that fundamentalness is unnecessary for using structural time series methods. Lippi and Reichlin (1993), Mertens and Ravn (2010), and Forni et al. (2017a,b) all contain examples in which non-invertible (and therefore non-fundamental) disturbances are identified using VARs. However, this paper is the first to propose a specific alternative condition to replace fundamentalness. Our theoretical analysis, therefore, provides a broader framework within which papers like these can be situated, alongside the many papers that do maintain the fundamentalness assumption. It also suggests new possibilities that may have been overlooked as the result of viewing fundamentalness as a methodological constraint that can only be circumvented in special cases.

While most of the literature on non-fundamentalness focuses on violations of invertibility, much less attention has been given to violations of causality. For example, Lippi and Reichlin (1994) excludes the possibility of non-causal disturbances by only considering non-invertible disturbances that can be obtained from fundamental disturbances by multiplying them with the inverse of a Blaschke matrix. Almost all of the subsequent literature has followed their lead. One exception is Lanne and Saikkonen (2013), which proposes one type of non-causal VAR model. However, that paper also assumes that disturbances are non-Gaussian, and exploits the information in higher-order moments. By contrast, we allow for non-causal disturbances while performing our analysis using only first and second moments.

We highlight additional points of comparison with the literature as they arise throughout the paper. As a supplementary reference, the online appendix also contains a list of frequently asked questions, which helps direct readers to the relevant discussions in the text.

The rest of the paper offers an application of the idea that structural disturbances only need to be recoverable to be identified. The application addresses two important empirical questions in macroeconomics. First, how important are (neutral) technological disturbances for explaining business cycle fluctuations? Second: by comparison,

a population concept.

how important are independent disturbances in expectations about technology?

The interesting thing about these questions is that it is not generally possible to answer them without allowing both for non-causality and non-invertibility. The reason for this is that economic agents might receive information about future technological changes in advance. If they do, then technological disturbances will generally affect observable variables before they occur, which means they will be non-causal. Furthermore, it isn't possible to determine, from current and past observables alone, whether current changes in expectations are actually due to subsequent technological developments. To do so, one would need to know what those subsequent developments are. This means that independent expectational disturbances must be non-invertible.

Nevertheless, as long as the econometrician has observations on technological change (as is typically assumed in the literature on technological “news”), both of these disturbances are generally recoverable. We use this insight to construct a new set of structural restrictions that correctly identify the disturbances of interest even when agents receive advance information of arbitrary form. These restrictions allow us to test the hypothesis that agents respond to technological developments in advance, and to provide quantitative estimates of the extent to which this is the case.

By comparison, most of the existing literature on technological news has not allowed either for non-invertible or non-causal disturbances. This is true, for example, in Beaudry and Portier (2006), Barsky and Sims (2011), and Barsky et al. (2015). The reason is that these papers equate technological disturbances with forecast errors about technology. However, forecast errors depend both on changes in technology and expectations, and therefore they confound the independent fluctuations in both. One exception is the work by Forni et al. (2017a,b), which allows for non-invertibility, but restricts the disturbances to be causal. Their procedure, however, can only identify technological and expectational disturbances in the presence of advance information of a very special type.

In an empirical example using postwar U.S. data, we find that technological disturbances are relatively unimportant for explaining business cycle fluctuations (12% of GDP). By contrast, purely expectational disturbances are much more important (69% of GDP). Our results are consistent with the view that technology is relatively more important at longer horizons, while shorter-run fluctuations are largely driven by independent changes in expectations about future technology. Our procedure also allows us to test the over-identifying restriction that these expectational disturbances

are consistent with agent rationality; we find little evidence to suggest that they are not.

2 Terms and conditions

This section defines the new concept of recoverability that we propose and presents a necessary and sufficient condition that can be used to check whether it is satisfied in any linear model. It also navigates through the thicket of other technical concepts that have been used in the literature. One challenge is that many of these concepts have very different economic meanings than their names suggest, mainly because they have been imported into the economic literature from other fields. Nevertheless, we opt to follow the current usage rather than propose entirely new names for the same concepts.

We begin by considering an arbitrary linear economic model mapping structural disturbances into observable variables,

$$y_t = \sum_{s=-\infty}^{\infty} \varphi_s \varepsilon_{t-s} \quad (1)$$

The n_ε dimensional process of structural disturbances, $\{\varepsilon_t\}$, is orthonormal white noise. In other words, its values have mean zero, unit variance, and are mutually uncorrelated. The sequence of $n_y \times n_\varepsilon$ matrices $\{\varphi_s\}$ has square summable components, so that the observable process $\{y_t\}$ is covariance stationary. We also assume that $\{y_t\}$ is linearly regular, which means that it does not have any purely deterministic components. All the linear models considered in the literature are special cases of this setup.

We now define several theoretical properties that the structural disturbances might have. It is important to note that all these are properties of structural models and not of data. To begin, if $\varphi_s = 0$ for all $s < 0$, then the model is said to have causal disturbances.

Definition 1. $\{\varepsilon_{k,t}\}$ is *causal* with respect to $\{y_t\}$ if $E[\varepsilon_{k,t} y_{l,s}] = 0$ for all l and $t > s$.

This definition says that the disturbance at time t cannot affect any observable variable before time t . This notion of causality is not the same as economic causality, and an important part of our argument will be that it should not be imposed a priori.

A common example of a model with causal disturbances is one with the state-space structure

$$y_t = Ax_t \quad \text{and} \quad x_t = Bx_{t-1} + C\varepsilon_t, \quad (2)$$

where x_t is an n_x dimensional state vector. In this case, $\varphi_s = AB^sC$ for all $s \geq 0$ and $\varphi_s = 0$ for all $s < 0$.

Now we are ready to define what we mean by recoverability. To do so, let $\mathcal{H}(y)$ denote the closed linear space spanned by the variables $y_{k,t}$ for all k, t . This space represents the information contained in the process $\{y_t\}$. Recoverability says that the value of the disturbance at each point in time can be perfectly inferred from this information; that is, from the past, present, and future values of the observables.⁴ In other words, it says that the “smoothed” variance of the disturbance of interest is zero in population.

Definition 2. $\{\varepsilon_{k,t}\}$ is *recoverable* from $\{y_t\}$ if $\varepsilon_{k,t} \in \mathcal{H}(y)$ for all t .

Recoverability is a weaker notion than invertibility, which has to do with whether the disturbances can be perfectly inferred only from the past and present (but not future) values of the observables. In other words, it says that the “filtered” variance of the disturbance of interest is zero in population. To define this concept formally, we let $\mathcal{H}_t(y)$ denote the closed linear space spanned by all the observable variables only up through time t .

Definition 3. $\{\varepsilon_{k,t}\}$ is *invertible* from $\{y_t\}$ if $\varepsilon_{k,t} \in \mathcal{H}_t(y)$ for all t .

Notice that invertibility does not imply causality, which means that it is possible in principle to infer non-causal disturbances from the current and past history of observables.⁵

All three of the properties we have defined so far have been stated in terms of a single scalar process $\{\varepsilon_{k,t}\}$. They can be extended to a multi-dimensional process if they apply to each of its constituent scalar processes individually. They can also be extended to any *representation* of the process $\{y_t\}$, such as the model in equation (1). For example, if $\{\varepsilon_{k,t}\}$ is causal for all $k = 1, \dots, n_\varepsilon$, then $\{\varepsilon_t\}$ is a causal n_ε dimensional process and equation (1) is a causal representation of $\{y_t\}$.

⁴Plagborg-Møller and Wolf (2018) independently adopt this same definition in their contemporaneous work on variance decompositions in linear projection instrumental variables models.

⁵This definition of invertibility follows Remark 6 on p. 88 of Brockwell and Davis (1991).

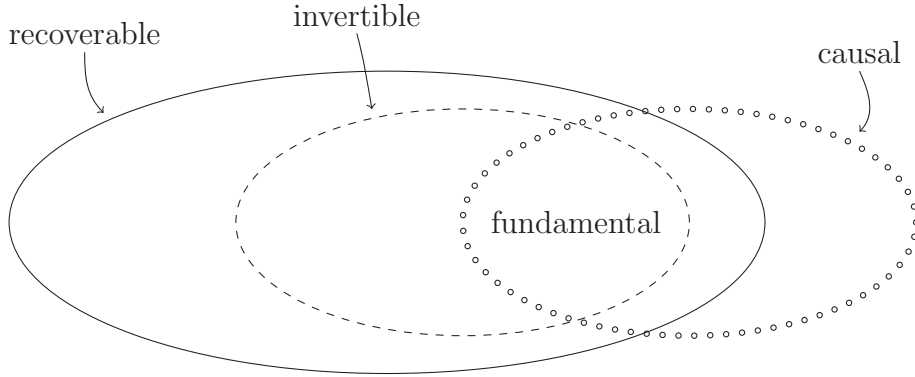


Figure 1: Relationships between different properties of disturbances.

Lastly, we turn to the notion of fundamentalness. As originally defined by Rozanov (1967), pp. 56-57, an orthonormal white noise process $\{\varepsilon_t\}$ is called fundamental if its values at time t form an orthonormal basis for the space $\mathcal{D}_t(y) \equiv \mathcal{H}_t(y) \ominus \mathcal{H}_{t-1}(y)$, which is the orthogonal complement of $\mathcal{H}_{t-1}(y)$ in $\mathcal{H}_t(y)$. Apart from being somewhat opaque, this definition has the awkward consequence of making fundamentalness asymmetrical with respect to causality and invertibility. This is because it implies that if a multi-dimensional process $\{\varepsilon_t\}$ is fundamental then none of its constituent processes $\{\varepsilon_{k,t}\}$ can be fundamental. To avoid this implication, we adopt a slightly modified definition.

Definition 4. $\{\varepsilon_{k,t}\}$ is *fundamental* with respect to $\{y_t\}$ if it is causal and invertible.

This definition coincides with Rozanov's original definition when it is applied to all the disturbances in a given representation. To see this, notice that if all the disturbances are causal, so that $\varphi_s = 0$ for all $s < 0$, then equation (1) implies that $\mathcal{H}_t(y) \subseteq \mathcal{H}_t(\varepsilon)$, where $\mathcal{H}_t(\varepsilon)$ is the space spanned by all disturbances up through time t . If all the disturbances are also invertible, then $\mathcal{H}_t(\varepsilon) \subseteq \mathcal{H}_t(y)$ as well. This means that $\mathcal{H}_t(\varepsilon) = \mathcal{H}_t(y)$, which, because the disturbances are orthonormal white noise, implies that the values $\varepsilon_{k,t}$ for $k = 1, \dots, n_\varepsilon$ form an orthonormal basis for $\mathcal{D}_t(y)$. It is easy to see that the converse is true as well. Figure (1) summarizes the relationships between recoverability, invertibility, fundamentalness, and causality in the form of a set diagram.

Our main interest is in determining the conditions under which the theoretical model in equation (1) predicts that the structural disturbances are recoverable. To

do so, it is helpful to introduce the $n_y \times n_\varepsilon$ dimensional function

$$\varphi(\lambda) = \sum_{s=-\infty}^{\infty} \varphi_s e^{-i\lambda s},$$

which is the (discrete) Fourier transform of the coefficient sequence $\{\varphi_s\}$. This function is defined for values of λ on the interval $[-\pi, \pi]$, and summarizes all the theoretical restrictions embedded in the economic model. For example, if the model has a state-space structure of the form (2), then this function takes the form $\varphi(\lambda) = A(I_{n_x} - B e^{-i\lambda})^{-1} C$. The benefit of working with this function is that it allows us to state a simple and intuitive condition that is both necessary and sufficient for recoverability. We first state the theorem and then provide a discussion. Its proof is in the appendix.

Theorem 1. $\{\varepsilon_{k,t}\}$ is recoverable from $\{y_t\}$ if and only if

$$\delta_k(I_{n_\varepsilon} - \varphi(\lambda)^\dagger \varphi(\lambda)) = 0$$

for almost all λ , where I_{n_ε} is the n_ε dimensional identity matrix, δ_k is the k -th row of I_{n_ε} , and \dagger denotes the Moore-Penrose pseudoinverse.

To understand the logic behind this result, it is helpful first to consider the case without dynamics. Suppose that $\varphi_s = 0$ for all $s \neq 0$, so that there is just a static relationship between the observable variables and the structural disturbances at each point in time, $y_t = \varphi_0 \varepsilon_t$, which means that $\varphi(\lambda) = \varphi_0$ is a constant matrix. The best estimate of ε_t based on the information in $y_{k,t}$ for $k = 1, \dots, n_y$ is given by the projection

$$\tilde{\varepsilon}_t = \varphi_0^\dagger y_t = \varphi_0^\dagger \varphi_0 \varepsilon_t.$$

This implies that the k -th disturbance is equal to its best estimate exactly when the k -th row of the matrix $\varphi_0^\dagger \varphi_0$ is equal to the k -th row of the identity matrix I_{n_ε} . In other words, when $\delta_k(I_{n_\varepsilon} - \varphi_0^\dagger \varphi_0) = 0$, which is the condition in the theorem. This same logic carries over directly to the more general dynamic case, with the only added proviso that the condition hold for almost all values of λ — that is, for all values of λ on $[-\pi, \pi]$ except possibly a set of Lebesgue measure zero.

If all the disturbances are recoverable, then $\varphi(\lambda)^\dagger \varphi(\lambda) = I_{n_\varepsilon}$, which implies that $\varphi(\lambda)$ has full column rank. Conversely, if $\varphi(\lambda)$ has full column rank, then the condition in Theorem (1) will be satisfied for all n_ε disturbances. By combining these observations, we arrive at the following corollary.

Corollary 1. $\{\varepsilon_t\}$ is recoverable from $\{y_t\}$ if and only if $\text{rank}[\varphi(\lambda)] = n_\varepsilon$ for almost all λ .

According to this corollary, a necessary condition for all the structural disturbances to be recoverable is that there be at least as many observable variables as disturbances, $n_y \geq n_\varepsilon$. The intuition is that it is not possible to identify n_ε separate sources of random variation without observations of at least n_ε random processes. In the more general case that we are only interested in recovering $m \leq n_\varepsilon$ of the disturbances, the necessary condition is that $n_y \geq m$.

For cases in which it is difficult to check the condition in Theorem (1) analytically, there is a simple numerical procedure that can be used. The linear regularity of $\{y_t\}$ implies that $\varphi(\lambda)$ has a constant rank for almost all λ , which means that it is possible to draw a number λ_u randomly from the interval $[-\pi, \pi]$ and numerically check which rows of $I_{n_\varepsilon} - \varphi(\lambda_u)^\dagger \varphi(\lambda_u)$ are zero vectors. In Matlab, an equivalent procedure is to execute the command

$$N = \text{null}(\varphi(\lambda_u)),$$

and check which rows of N are zero vectors. If N is an empty matrix, then $\varphi(\lambda_u)$ is full column rank, in which case all the disturbances are recoverable.

Example 1. This example demonstrates how to check recoverability numerically. Consider the following dynamic system with three disturbances and three observable variables,

$$y_{1,t} = -0.490\varepsilon_{1,t-1} - 0.784\varepsilon_{1,t-2} + 0.098\varepsilon_{2,t-2} + 0.120\varepsilon_{3,t} + 0.496\varepsilon_{3,t-1}$$

$$y_{2,t} = -0.500\varepsilon_{1,t} - 0.800\varepsilon_{1,t-1} + 0.100\varepsilon_{2,t-1} + 0.200\varepsilon_{3,t}$$

$$y_{2,t} = 0.400\varepsilon_{1,t-1} + 0.640\varepsilon_{1,t-2} - 0.080\varepsilon_{2,t-2} - 0.200\varepsilon_{3,t} - 0.660\varepsilon_{3,t-1}$$

The associated function $\varphi(\lambda)$ takes the form

$$\varphi(\lambda) = \begin{bmatrix} -0.490e^{-i\lambda} - 0.784e^{-2i\lambda} & 0.098e^{-2i\lambda} & 0.120 + 0.496e^{-i\lambda} \\ -0.500 - 0.800e^{-i\lambda} & 0.100e^{-i\lambda} & 0.200 \\ 0.400e^{-i\lambda} + 0.640e^{-2i\lambda} & -0.080e^{-2i\lambda} & -0.200 - 0.660e^{-i\lambda} \end{bmatrix}$$

To check the condition in Theorem (1), we randomly draw $\lambda_u = 0.109$ from a uniform distribution over $[-\pi, \pi]$ and execute

$$N = \text{null}(\varphi(\lambda_u)) = \begin{bmatrix} -0.0768 + 0.0000i \\ -0.9962 - 0.0418i \\ 0.0000 + 0.0000i \end{bmatrix}.$$

The third row of N is zero, which tells us that $\{\varepsilon_{3,t}\}$ is recoverable but the other two disturbances are not. The reason for this is that each element of the second column of $\varphi(\lambda)$ is equal to the corresponding element of the first column multiplied by the factor

$$\frac{-0.100e^{-i\lambda}}{0.500 + 0.800e^{-i\lambda}}.$$

So, the first two disturbances cannot be inferred because the patterns of fluctuations they generate in the observable variables are not dynamically independent. \triangle

We conclude this section by providing necessary and sufficient conditions for both causality and invertibility; together these represent necessary and sufficient conditions for fundamentalness. These conditions are only of tangential importance for our purposes in this paper, but will be helpful for diagnosing situations in which disturbances are recoverable but either non-causal or non-invertible. Because they apply to any linear model, they are also slightly more general than those typically presented in the literature.

The causality condition follows from a straightforward application of the definition. It just says that the Fourier coefficients of $\varphi(\lambda)$ associated with negative powers of $e^{-i\lambda}$ must be zero for the disturbance of interest. Regarding notation, we use an asterisk to denote complex conjugate transposition.

Theorem 2. $\{\varepsilon_{k,t}\}$ is causal with respect to $\{y_t\}$ if and only if

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda s} \delta_k \varphi(\lambda)^* d\lambda = 0 \quad \text{for all } s < 0.$$

The condition for invertibility is somewhat more involved, and requires us to make use of Wold's decomposition theorem (see, e.g. Rozanov, 1967, p. 56). This theorem implies that the observables $\{y_t\}$ can always be represented as a one-sided moving average

$$y_t = \sum_{s=0}^{\infty} \gamma_s w_{t-s}, \tag{3}$$

where $\{w_t\}$ is an n_w dimensional orthonormal white noise process which is fundamental with respect to $\{y_t\}$. As we did for the structural model, we can define the function $\gamma(\lambda)$ as the discrete Fourier transform of the coefficients in this representation,

$$\gamma(\lambda) = \sum_{s=0}^{\infty} \gamma_s e^{-i\lambda s}.$$

With this function, we can state the following theorem.

Theorem 3. $\{\varepsilon_{k,t}\}$ is invertible with respect to $\{y_t\}$ if and only if it is recoverable with respect to $\{y_t\}$ and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda s} \delta_k \varphi(\lambda)^\dagger \gamma(\lambda) d\lambda = 0 \quad \text{for all } s < 0.$$

3 Motivating example

Why might it be economically interesting to consider recoverable but non-invertible and non-causal disturbances? This section presents an example. Suppose that an econometrician is interested in separately determining the importance of disturbances to technology and independent disturbances to expectations about future technology. The basic idea is that if economic agents happen to receive information about future technological disturbances in advance, it is not generally possible to accomplish this task without relaxing both causality and invertibility.

To see why this is the case, let us consider a simple but general information structure in which agents receive advance information about future technology.⁶ The level of technology at time t is the accumulation of current and past technological disturbances,

$$a_t = \sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-k}^a,$$

where the sequence of weights $\{\alpha_k\}$ determines the dynamics of technology. At each time t , in addition to observing the current and past values of a_t , agents also receive a signal s_t that contains additional information about future technological disturbances, but is contaminated by an independent noise process,

$$s_t = \sum_{k=1}^{\infty} \varsigma_k \varepsilon_{t+k}^a + v_t, \quad v_t = \sum_{k=0}^{\infty} \nu_k \varepsilon_{t-k}^v.$$

Apart from being square-summable, the sequences $\{\alpha_k\}$, $\{\varsigma_k\}$, and $\{\nu_k\}$ are arbitrary.

Agents cannot separately observe the noise process, which means they do not know whether changes in the signal at time t reflect actual future developments in technology or unrelated noise. Nevertheless, because the signal is informative, it is rational for agents to rely on it when forming their expectations of future technology.

⁶We focus on technology and expected technology, but this example can apply equally well to other primitive driving processes (e.g. the non-systematic components of monetary or fiscal policy).

As a result, agents' expectations and actions (insofar as they are forward looking) are affected both by future technological disturbances and the independent disturbances in the unobservable noise process. This means that the technological disturbances, $\{\varepsilon_t^a\}$, are inherently non-causal with respect to agents' actions, while the independent disturbances to agents' expectations, $\{\varepsilon_t^v\}$, are inherently non-invertible.⁷

To the extent that causality and invertibility are viewed as necessary conditions for using time series models like VARs, their inherent violation seems to pose a serious problem. For this reason, some have concluded that such models simply cannot be used, and a more fully structural empirical strategy is required. In the following section, we describe a proposal that avoids this conclusion. Specifically, we suggest that as long as the disturbances are recoverable, it is still feasible to identify them using, for example, a VAR. We will discuss the proposal in more detail below, but before doing so it makes sense to ask whether the disturbances in this example are recoverable.

Despite the inherent violations of causality and invertibility that advance information creates, it is easy to see that both disturbances are still recoverable with respect to agents' information. The two variables observed by agents are related to the structural disturbances by a linear relationship of the form

$$\begin{bmatrix} a_t \\ s_t \end{bmatrix} = \sum_{k=-\infty}^{\infty} \begin{bmatrix} \alpha_k & 0 \\ \varsigma_{-k} & \nu_k \end{bmatrix} \begin{bmatrix} \varepsilon_{t-k}^a \\ \varepsilon_{t-k}^v \end{bmatrix},$$

where $\alpha_k = \nu_k = 0$ for all $k < 0$ and $\varsigma_k = 0$ for all $k < 1$. Letting $\varphi(\lambda)$ denote the Fourier transform of the sequence of matrix coefficients in this equation, we have

$$|\varphi(\lambda)| = \alpha(\lambda)\nu(\lambda),$$

where $\alpha(\lambda)$ and $\nu(\lambda)$ are the Fourier transforms of $\{\alpha_k\}$ and $\{\nu_k\}$, respectively. Under the natural assumption that the technological disturbances are recoverable from technology and the non-technology disturbances are recoverable from the noise process, it follows from Corollary (1) that $\alpha(\lambda)$ and $\nu(\lambda)$, and so also the product of the two, are nonzero for almost all values of λ . This implies that $\varphi(\lambda)$ has full column rank, which indicates that both disturbances are recoverable from agents' observables.

⁷The point that signal-extraction problems of this type generate inherent non-invertibility has been made by Blanchard et al. (2013).

Of course, it might seem implausible to suppose that an econometrician also observes the signal process, especially if the signal is interpreted as a way of representing agents' imperfect perceptions of future conditions, rather than a literal piece of publicly available data. Fortunately, both disturbances are still recoverable using agents' expectations instead. To see this, let us define the expectational variable $b_t = E_t[a_{t+h}]$, for arbitrary $h > 0$. Then we can write

$$\begin{bmatrix} a_t \\ b_t \end{bmatrix} = \sum_{k=-\infty}^{\infty} \begin{bmatrix} \alpha_k & 0 \\ \phi_k^a & \phi_k^v \end{bmatrix} \begin{bmatrix} \varepsilon_{t-k}^a \\ \varepsilon_{t-k}^v \end{bmatrix}, \quad (4)$$

where the coefficients $\{\phi_k^a, \phi_k^v\}$, which are implicitly functions of the other model coefficients, solve the linear prediction problem of projecting a_{t+h} on the space spanned by observables up through time t . The determinant of the associated function $\varphi(\lambda)$ is given by

$$|\varphi(\lambda)| = \alpha(\lambda)\phi^v(\lambda),$$

where $\phi^v(\lambda)$ is the Fourier transform of $\{\phi_k^v\}$. Apart from pathological cases (e.g. the agents place zero weight on the signal), it follows immediately from the fact that $\alpha(\lambda)$ and $\nu(\lambda)$ are nonzero for almost all λ that $\phi^v(\lambda)$ will be as well. Therefore, both disturbances continue to be recoverable from the variables which are either directly or indirectly observable to the econometrician.

This example is meant to illustrate why it might be interesting to entertain disturbances which are recoverable even if they are not causal or invertible. We now explain our general proposal to view recoverability as the key feasibility condition for using structural time series methods to identify disturbances. Throughout the discussion it may be helpful to keep this example in mind for concreteness. We return to it in section (6), where we work through a specific application of our proposal.

4 Relation to VARs

This section explains our proposal to view recoverability — not fundamentalness — as the appropriate feasibility condition for identifying structural disturbances from an unstructured time series model. The time series model that is most commonly used in macroeconomics, following the proposal by Sims (1980), is the vector autoregression (VAR). For the sake of concreteness, it will be convenient for us to frame our discussion in terms of this specific time series model. However, we will make clear as we go along

that nothing about the general strategy is tied to VARs, and other time series models could be used as well (e.g. models with moving average terms).

According to most treatments, VAR identification of structural disturbances entails at least the following two assumptions:⁸

- (a) The observables $\{y_t\}$ can be adequately represented by a stable VAR model of the form

$$y_t = B_1 y_{t-1} + \dots + B_p y_{t-p} + u_t,$$

where $\{u_t\}$ is a white noise process with covariance matrix Σ .

- (b) In the candidate theoretical model(s) of interest, each structural disturbance of interest $\{\varepsilon_{k,t}\}$ has values which can be expressed as linear combinations of one-step-ahead forecast errors

$$\varepsilon_{k,t} = D_k(y_t - E[y_t | \mathcal{H}_{t-1}(y)]),$$

where D_k is a constant vector.

Taken together, these assumptions suggest the following two-step empirical procedure. First, use statistical techniques to fit the VAR model in assumption (a) to the data. Second, use economic theory to select the appropriate vector D_k in assumption (b) and then identify the structural disturbances as $\varepsilon_{k,t} = D_k u_t$. The advantage of this approach is that only a subset of the theoretical restrictions from the candidate model (or class of models) is needed. Its empirical conclusions regarding the structural disturbances can therefore be interpreted as robust across the range of different models that are consistent with those restrictions.

Notice that assumption (b) does not say anything about specifically which theoretical restrictions should be used. Rather, it just amounts to the requirement that the structural disturbances must be both causal and invertible. They must be causal because $y_{l,s} \in \mathcal{H}_{t-1}(y)$ for any $s < t$, so

$$E[\varepsilon_{k,t} y_{l,s}] = D_k E[(y_t - E[y_t | \mathcal{H}_{t-1}(y)]) y_{l,s}] = 0,$$

and they must be invertible because

$$y_{k,t} - E[y_{k,t} | \mathcal{H}_{t-1}(y)] \in \mathcal{H}_t(y)$$

⁸See, e.g., Kilian and Lutkepohl (2017) or Stock and Watson (2016).

for all k , so $\varepsilon_{k,t} \in \mathcal{H}_t(y)$ as well. The implication is that a failure of either one of these conditions poses a problem for the validity of VAR-based identification of structural disturbances.

By contrast, our proposal is that the problem posed by non-fundamentalness can be avoided by replacing assumption (b) above with

(b') In the candidate theoretical model(s) of interest, each structural disturbance of interest $\{\varepsilon_{k,t}\}$ is recoverable with respect to $\{y_t\}$.

In other words, we suggest removing the purely methodological constraint that the only admissible theoretical restrictions are ones that can be articulated as linear combinations of contemporaneous one-step-ahead forecast errors. The only real requirement to identify the structural disturbances is that the observable variables available to the econometrician contain enough information for the disturbances to be inferred, which is the content of assumption (b').

We believe this proposal has several advantages, five of which we discuss here. First, it expands the feasible set of theoretical restrictions that can be entertained in the economic step of the analysis. If there are good economic reasons to impose fundamentalness, of course this option is still available. But under our proposal there is no problem associated with imposing alternative restrictions instead. Like assumption (b), assumption (b') does not say anything about specifically which theoretical restrictions should be used. Under either assumption, the appropriate restrictions will depend on the theory and the type of disturbances the researcher has in mind. The difference is that (b') does not entail a pre-commitment to imposing restrictions only of a particular type.

Second, it clarifies that the appropriateness of using VARs is unrelated to the question of whether the structural disturbances are fundamental. This is because the only purpose of the VAR is to summarize the dynamic relationships among the observable variables. In other words, it is used as a flexible tool for estimating autocovariances.⁹ Whether or not this is a good idea is really a statistical issue, and statistical criteria for evaluating goodness of fit can and should be used to guide this part of the analysis. In some cases, it may be that alternative unstructured time

⁹This is consistent with the original proposal of Sims (1980): “The style I am suggesting we emulate is that of frequency-domain time series theory..., in which what is being estimated (e.g., the spectral density) is implicitly part of an infinite-dimensional parameter space...” (p. 15).

series models are preferred. In those cases, assumption (a) can be modified to reflect the time series model that makes the best statistical sense. But the point is that a rejection of assumption (b) does not require a rejection of assumption (a).

Third, it clarifies exactly when non-invertibility is an omitted variables problem. Non-invertibility means that disturbances at time t cannot be identified using data up to time t . This identification problem can occur for two reasons. The first is about variables: the variables needed to identify the disturbances are omitted from the econometrician’s dataset.¹⁰ The second is about timing: the econometrician has the right variables, but needs to use their values at dates later than t to identify the time- t disturbances. If the disturbances are recoverable, then the problem is about timing, not omitted variables. This distinction becomes especially important in situations (such as the example in the previous section) when the disturbances are inherently non-invertible with respect to any realistic set of observables.

Fourth, it reduces the emphasis placed on one-step-ahead forecast errors. Most papers require disturbances to be linear combinations of current forecast errors, while some have suggested that linear combinations of current and future forecast errors should be allowed as well.¹¹ The assumption that the disturbances are recoverable of course implies that there must be *some* relationship between them and the forecast errors, but it does not restrict what that relationship should look like. Forecast errors are just transformations of the observables, and one could easily imagine many other transformations (e.g. backcast errors). Though forecast errors have been given special attention in the literature, they are not — and should not be — inherently tied to structural disturbances.

Fifth, it incorporates recent advances in VAR-based identification. Papers by Mertens and Ravn (2010) and Forni et al. (2017a,b), for example, combine VARs with theoretical restrictions to analyze structural disturbances that are causal but not invertible. We add to this literature in our application below, where we are interested in identifying a set of disturbances that is neither causal nor invertible. Instead of appearing unusual or non-standard because they are inconsistent with assumption (b), these applications fit in naturally alongside the rest of the existing

¹⁰This is the reason discussed by Alessi et al. (2011), Forni and Gambetti (2014), Kilian and Lütkepohl (2017), Nakamura and Steinsson (2018), and Beaudry et al. (2019), among others.

¹¹Bernanke (1986) and Blanchard and Watson (1986) only allow current forecast errors, while Lippi and Reichlin (1994) only allow current and future forecast errors.

VAR literature under assumption (b').

In addition to these advantages, there are also a few other aspects of our proposal that deserve comment. One is the question of whether it is correct to think that our proposal requires a greater reliance on economic theory.¹² The answer is that it does not, once we think of fundamentalness itself as an economic restriction. The identification of non-fundamental disturbances entails *relaxing* the economic restriction of fundamentalness and imposing alternative restrictions in its place. The fact that these alternative restrictions are different does not imply that they are stronger. Indeed, it is possible that they could even be weaker from an economic perspective, as we will suggest is the case in our application below.

Another question is the extent to which our proposal is relevant to debates about the appropriateness of fitting one fully specified time series model in the statistical step of the empirical procedure described above.¹³ The answer is that our proposal is conceptually independent of this debate. As we have mentioned above, the usefulness of the time series model (e.g. the VAR) is as a tool for estimating the autocovariances of the observable variables. Even if it is preferable to estimate these autocovariances using alternative time series methods in certain situations, the question still arises: should we only try to identify fundamental disturbances, and if not, is there some alternative concept that tells us which disturbances cannot be identified? Our proposal addresses this question.

Lastly, it is worth pointing out that it is possible to weaken our assumption (b') at the cost of being able to say less about the disturbances of interest. This assumption is necessary for identifying the disturbances, and it implies that all other objects of economic interest related to these disturbances are identified as well: impulse responses, variance shares, time decompositions, etc. However, if the objective were only to identify some of these objects of interest (as in the literature on external instruments; e.g. Stock and Watson, 2018), or only to set identify them (as in Plagborg-Møller and Wolf, 2018), then it would be possible to consider disturbances that are not recoverable. Nevertheless, much of the structural time series literature on set identification, such as the literature on sign identification (cf. Uhlig, 2017), operates under the assumption that the underlying structural disturbances are fundamental.

¹²Cf. the discussion on p. 597 of Kilian and Lütkepohl (2017).

¹³See, e.g. Jordà (2005) and Nakamura and Steinsson (2018).

5 VAR procedure

This section implements our proposal and shows how to use VAR methods to analyze the two types of structural disturbances from the example in section (3). The fact that both disturbances are generally recoverable from observations of technology and expectations suggests a natural set of structural restrictions that can be used in a VAR analysis. The first is that technological disturbances are the fundamental disturbances in technology. As long as technology is observable, technological disturbances can be identified without any reference to agents' beliefs about technology. The second is that the independent disturbances to expectations are the fundamental disturbances in the part of expectations that is independent of technology at all leads and lags. In the information structure above, the only reason that expectations about technology fluctuate apart from actual fluctuations in technology is because of the independent fluctuations generated by the noise process. To summarize, we can make these identifying assumptions:

- (i) $\{\varepsilon_t^a\}$ are the fundamental disturbances in $\{a_t\}$ such that $E[a_t\varepsilon_t^a] \geq 0$, and
- (ii) $\{\varepsilon_t^v\}$ are the fundamental disturbances in $\{b_t - E[b_t|\mathcal{H}(a)]\}$ such that $E[(b_t - E[b_t|\mathcal{H}(a)])\varepsilon_t^v] \geq 0$.

These restrictions amount to imposing the following “zeros” in the mapping from structural disturbances to observables,

$$\begin{bmatrix} a_t \\ b_t \end{bmatrix} = \dots + \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{t+1}^a \\ \varepsilon_{t+1}^v \end{bmatrix} + \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \begin{bmatrix} \varepsilon_t^a \\ \varepsilon_t^v \end{bmatrix} + \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \begin{bmatrix} \varepsilon_{t-1}^a \\ \varepsilon_{t-1}^v \end{bmatrix} + \dots,$$

which is consistent with the theoretical model in equation (4). The sign restrictions in (i) and (ii) are included because fundamental disturbances in scalar processes are only unique up to a sign change. They reflect the normalization that a positive technological disturbance generates a positive response in technology on impact, with the analogous normalization for the expectational disturbances.

Before describing the details of how we incorporate these restrictions into a VAR analysis, we pause to discuss three ways that these restrictions are related to others in the existing literature. First, the technological disturbances identified by restriction (i) are not forecast errors in technology with respect to all the observables. A common

practice in the VAR literature is to associate the technological disturbance at time t with some version of the forecast error

$$w_t^a = a_t - E_{t-1}[a_t] \quad (5)$$

The problem is that these forecast errors are affected both by disturbances in technology and non-technological disturbances in expectations. Since we are interested in disentangling these two types of disturbances it would not make sense for us to follow this practice. Instead, we follow Basu et al. (2006) and identify the technological disturbance from technology only.

Second, the expectational disturbances identified by restriction (ii) are not forecast errors in expectations that are orthogonal to forecast errors in technology. The macroeconomic VAR literature on “news” follows the practice of identifying technological disturbances according to equation (5), and then builds on this by identifying expectational disturbances as proportional to the orthogonalized forecast error

$$w_t^b = (b_t - E_{t-1}[b_t]) - E[(b_t - E_{t-1}[b_t])|w_t^a] \quad (6)$$

Like the forecast errors in equation (5), these orthogonalized forecast errors also mix technological and non-technological disturbances. They do not identify variation in expectations that are independent of technology, making them inappropriate for assessing the independent causal role of expectations. This point is discussed in greater detail in Chahrour and Jurado (2018).

Third, restriction (ii) correctly identifies the expectational disturbances for any information structure in the general class described above. A recent paper by Forni et al. (2017b) aims to identify noise in agents’ signals about future technology. At one point, they observe that a natural restriction for identifying such noise is the requirement that technology “is not affected by noise at any lag” (p. 130). However, in their subsequent analysis, they do not use this restriction. Instead, they propose an alternative multi-step algorithm which is only able to correctly identify the signal noise in the special case of the above information structure where $\varsigma_k = 0$ for all k but one. By contrast, restriction (ii) has the advantage of both being more transparent and more general, in the sense that it works for any sequence of weights $\{\varsigma_k\}$.

To demonstrate how restrictions (i) and (ii) can be used as part of a VAR analysis, let $\{y_t\}$ denote the n_y dimensional process observed by the econometrician. Restriction (i) requires the econometrician to observe technology, so a_t is included

as an element of y_t . For now, we will continue to suppose that the observable process is covariance stationary and linearly regular. It is straightforward to allow for different forms of non-stationarity, as we do in practice; however, explicitly doing so at this point would unnecessarily complicate the discussion. Restriction (ii) requires the econometrician to observe expectations of future technology. To meet this requirement, we follow the common practice in the VAR literature of equating agents' expectations with optimal econometric forecasts. Therefore, we assume that $b_t = E_t[a_{t+h}] = E[a_{t+h}|\mathcal{H}_t(y)]$ for any t and h . We should note that this assumption is only necessary for identifying the expectational disturbances, and not the technological ones.

The first stage of the analysis involves fitting an unstructured VAR model of the form

$$y_t = B_1 y_{t-1} + \dots + B_p y_{t-p} + u_t,$$

where $\{u_t\}$ is a white noise process with covariance matrix Σ . The purpose of this step is just to characterize the autocovariances; by using a VAR we are assuming that this type of model is adequate for that purpose. The autocovariances implied by this VAR are conveniently summarized by the spectral density, which in this case takes the form

$$f_y(\lambda) = \frac{1}{2\pi} (I_{n_y} - B_1 e^{-i\lambda} - \dots - B_p e^{-ip\lambda})^{-1} \Sigma [(I_{n_y} - B_1 e^{-i\lambda} - \dots - B_p e^{-ip\lambda})^{-1}]^*.$$

Taking this spectral density as given, the second stage of the analysis is to use restrictions (i) and (ii) to identify the structural disturbances of interest. To do this, let us focus on the joint dynamics of technology and expectations. Letting δ_a denote the $1 \times n_y$ constant vector such that $a_t = \delta_a y_t$, and $b_t = \sum_{s=0}^{\infty} \beta_s y_{t-s}$ the optimal forecast of a_{t+h} implied by the VAR, the joint spectral density of $\{a_t\}$ and $\{b_t\}$ is given by

$$f(\lambda) = \begin{bmatrix} f_a(\lambda) & f_{ab}(\lambda) \\ f_{ba}(\lambda) & f_b(\lambda) \end{bmatrix} = \begin{bmatrix} \delta_a \\ \beta(\lambda) \end{bmatrix} f_y(\lambda) \begin{bmatrix} \delta'_a & \beta(\lambda)^* \end{bmatrix},$$

where $\beta(\lambda)$ is the Fourier transform of $\{\beta_s\}$. Now we need to obtain a unique factorization of $f(\lambda)$ of the form

$$f(\lambda) = \frac{1}{2\pi} \varphi(\lambda) \varphi(\lambda)^*, \quad (7)$$

where the factor $\varphi(\lambda)$ is the one defined by our restrictions. The Fourier coefficients of $\varphi(\lambda)$ then define the linear relationship from the structural disturbances to technology

and expectations,

$$\begin{bmatrix} a_t \\ b_t \end{bmatrix} = \sum_{s=-\infty}^{\infty} \varphi_s \begin{bmatrix} \varepsilon_t^a \\ \varepsilon_t^v \end{bmatrix}.$$

To solve our factorization problem, we begin by observing that one implication of restriction (ii) is that $\varphi(\lambda)$ must have a lower-triangular form,

$$\varphi(\lambda) = \begin{bmatrix} \varphi_{11}(\lambda) & 0 \\ \varphi_{21}(\lambda) & \varphi_{22}(\lambda) \end{bmatrix}.$$

This reflects the assumption that the expectational disturbances are independent of technology. Substituting this expression for $\varphi(\lambda)$ into equation (7), we have the system

$$\begin{bmatrix} f_a(\lambda) & f_{ab}(\lambda) \\ f_{ba}(\lambda) & f_b(\lambda) \end{bmatrix} = \frac{1}{2\pi} \begin{bmatrix} |\varphi_{11}(\lambda)|^2 & \varphi_{11}(\lambda)\overline{\varphi_{21}(\lambda)} \\ \varphi_{21}(\lambda)\overline{\varphi_{11}(\lambda)} & |\varphi_{21}(\lambda)|^2 + |\varphi_{22}(\lambda)|^2 \end{bmatrix}.$$

First, consider the upper-left equation. Restriction (i) says that $\varphi_{11}(\lambda)$ can be obtained from $f_a(\lambda)$ by finding the version of Wold's decomposition of $\{a_t\}$ with a non-negative leading coefficient. The coefficients in this decomposition are unique and can be computed using standard procedures (e.g. Rozanov (1967) pp. 45-47). Next, the lower-left equation uniquely determines $\varphi_{21}(\lambda)$ as a function of $f_{ba}(\lambda)$ and $\varphi_{11}(\lambda)$, the second of which has already been determined from the upper-left equation. Lastly, the lower-right equation says that

$$\frac{1}{2\pi}|\varphi_{22}(\lambda)|^2 = f_b(\lambda) - \frac{1}{2\pi}|\varphi_{21}(\lambda)|^2.$$

By restriction (ii), $\varphi_{22}(\lambda)$ is uniquely determined from Wold's decomposition of the process with spectral density $f_b(\lambda) - \frac{1}{2\pi}|\varphi_{21}(\lambda)|^2$. This can be computed in the same manner as $\varphi_{11}(\lambda)$. Therefore, we have shown that the factor $\varphi(\lambda)$ is unique and how to obtain it from the unstructured VAR coefficients.

Given $\varphi(\lambda)$, the structural disturbances $\varepsilon_t = (\varepsilon_t^a, \varepsilon_t^v)'$ can be recovered from the observables via a linear transformation of the form

$$\varepsilon_t = \sum_{s=-\infty}^{\infty} \psi_s y_{t-s},$$

where $\{\psi_s\}$ are the Fourier coefficients of

$$\psi(\lambda) = \varphi(\lambda)^{-1} \begin{bmatrix} \delta_a \\ \beta(\lambda) \end{bmatrix}.$$

Using this mapping from observables to structural disturbances, it is possible to compute other objects of interest as well. For example, the response of $y_{k,t+s}$ to a unit impulse in $\varepsilon_{l,t}$ is given by

$$\text{IR}_{k,l}(s) = \int_{-\pi}^{\pi} e^{i\lambda s} \delta_k f_y(\lambda) \psi(\lambda)^* \delta_l' d\lambda. \quad (8)$$

Or, the share of the variance in the process $\{y_{k,t}\}$ due to the disturbance $\{\varepsilon_{l,t}\}$ over the frequency range $\Delta = [\lambda_1, \lambda_2]$ is given by

$$\text{VS}_{kl}(s) = \int_{\Delta} |\delta_k f_y(\lambda) \psi(\lambda)^* \delta_l'|^2 d\lambda \left(\int_{\Delta} f_{y,kk}(\lambda) d\lambda \right)^{-1}. \quad (9)$$

6 Empirical example

As an empirical example, we now use restrictions (i) and (ii) from the previous section to evaluate the importance of disturbances to technology and expectations in postwar U.S. data. The analysis uses seven quarterly macroeconomic variables from 1948:Q1-2018:Q4: a measure of technology, real gross domestic product (GDP), real consumption of non-durable goods and services, hours worked in the non-farm business sector, an index of real stock prices, inflation in the GDP deflator, and the nominal interest rate on 3-month Treasury bills. The measure of technology is utilization-adjusted total factor productivity (TFP) (Basu et al., 2006), the stock price index is the quarterly NYSE/AMEX/NASDAQ value-weighted index from CRSP, and all other variables are taken from FRED. Real variables are constructed from their nominal counterparts using the GDP deflator, and all variables except inflation and the interest rate are used in natural logarithms.

We use ordinary least squares to estimate a fourth-order VAR on the levels of the seven macroeconomic variables. To allow for the possibility that technology may be integrated of order one, we modify restrictions (i) and (ii) in the following way: letting a_t denote the natural logarithm of TFP and $b_t = E_t[a_{t+h}]$,

- (i) $\{\varepsilon_t^a\}$ are the fundamental disturbances in $\{\Delta a_t\}$ such that $E[\Delta a_t \varepsilon_t^a] \geq 0$, and
- (ii) $\{\varepsilon_t^v\}$ are the fundamental disturbances in $\{\Delta b_t - E[\Delta b_t | \mathcal{H}(\Delta a)]\}$ such that $E[(\Delta b_t - E[\Delta b_t | \mathcal{H}(\Delta a)]) \varepsilon_t^v] \geq 0$.

Lastly, since $b_t = E_t[a_{t+h}]$ is an h -period ahead forecast, we need to specify the forecast horizon h that we use to define this variable. Because we are interested in business-cycle fluctuations, we set $h = 20$, which is roughly the midpoint of the conventional business-cycle range of 6 to 32 quarters.

Before applying the procedure describe above to actual data, we examine its performance in a Monte Carlo exercise. The exercise is based on a structural model that includes foresight of the form in section (3) and which been estimated by Blanchard et al. (2013) and Chahrour and Jurado (2018). In the appendix, we show that (i) the structural disturbances are recoverable for our theoretical data generating process; (ii) the procedure recovers structural impulse responses nearly perfectly in long samples; and (iii) the procedure does a good job recovering impulse responses in finite samples.¹⁴

Figure (2) displays the estimated impulse response coefficients of a_t and b_t to ε_t^a and ε_t^v .¹⁵ The top left panel of the figure plots the response of technology a_t to a technological disturbance which occurs at time $t = 0$. It shows that technology is close to being a random walk, as other papers have found. The bottom left panel plots the response of expected technology b_t to a technological disturbance. It shows a small but statistically significant response in anticipation of this disturbance (values to the left of zero), with a peak response around one year in advance. This indicates that agents can at least partially anticipate future developments in technology.

The upper right panel of Figure (2) shows the response of technology to the purely expectational disturbance, which is zero at all horizons, consistent with the identifying assumption that these disturbances are independent of technology. Finally, the lower right panel shows that expected technology increases in response to the expectational disturbance, with an effect that gradually declines over time. This gradual decline is consistent with the standard effects of noise shocks under rational expectations.

Figure (3) plots the responses of the other observable variables to the same two disturbances. The first column shows that GDP, consumption, and hours all exhibit

¹⁴For comparison, we have also performed this exercise using the procedure developed by Forni et al. (2017b). We find that this procedure is unable to recover the structural responses to either disturbance regardless of the sample size, even in the best-case scenario that the econometrician can directly observe the latent signal from the model. These results are included in the online replication package.

¹⁵These are computed by cumulating the responses of Δa_t and Δb_t from $t = -10$ to $t = 20$. The impulse response function is defined in (8).

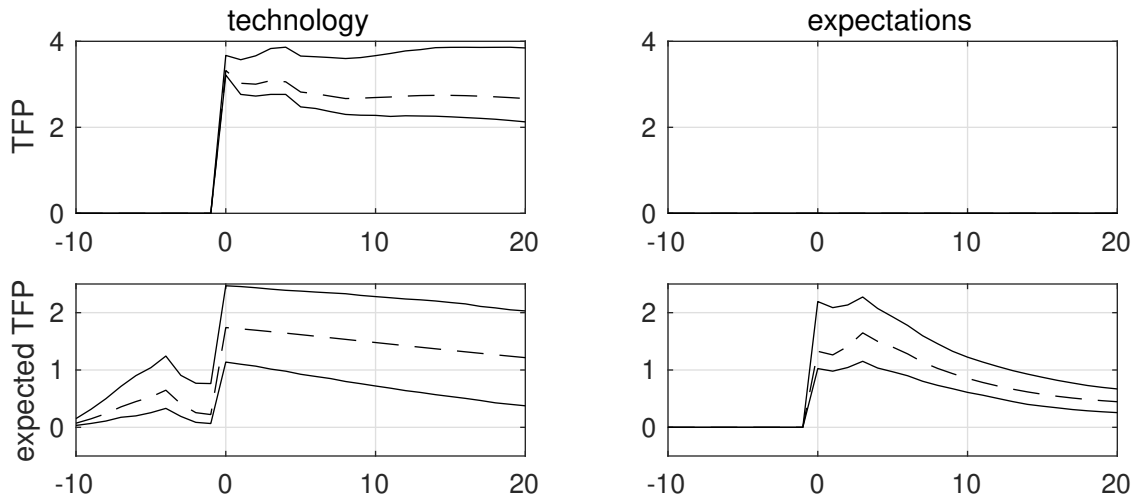


Figure 2: Impulse responses estimated from U.S. data. These figures display how TFP and expected TFP respond to a one standard deviation impulse in the disturbance to technology (left column) and the independent disturbance to expectations (right column) at time $t = 0$. The dashed lines are the point estimates and the solid lines represent the 90% bias-corrected bootstrap confidence interval. All units are annualized log points.

modest but statistically significant increases in anticipation of the technological disturbance. Stock prices also increase, while inflation and interest rates are essentially unchanged. Contemporaneously with the realization of the technological disturbance at time zero, hours fall substantially, while GDP and consumption are essentially unchanged, consistent with the findings of Basu et al. (2006). After the disturbance occurs, GDP and consumption gradually rise, consistent with technological disturbances having persistent macroeconomic effects.

The second column of Figure (3) plots the responses of the same endogenous variables to the expectational disturbance. This disturbance generates positive comovement between GDP, consumption, hours, and stock prices. These variables all increase contemporaneously with the disturbance and subsequently evolve in a hump-shaped manner. For all variables other than stock prices, impulse responses are small and insignificant in the time periods prior to the disturbance. While our identifying assumptions require this disturbance to affect expected technology only starting at time zero, the fact that we also see essentially no advance responses in the other variables is a result. This result is consistent with a rational expectations interpretation

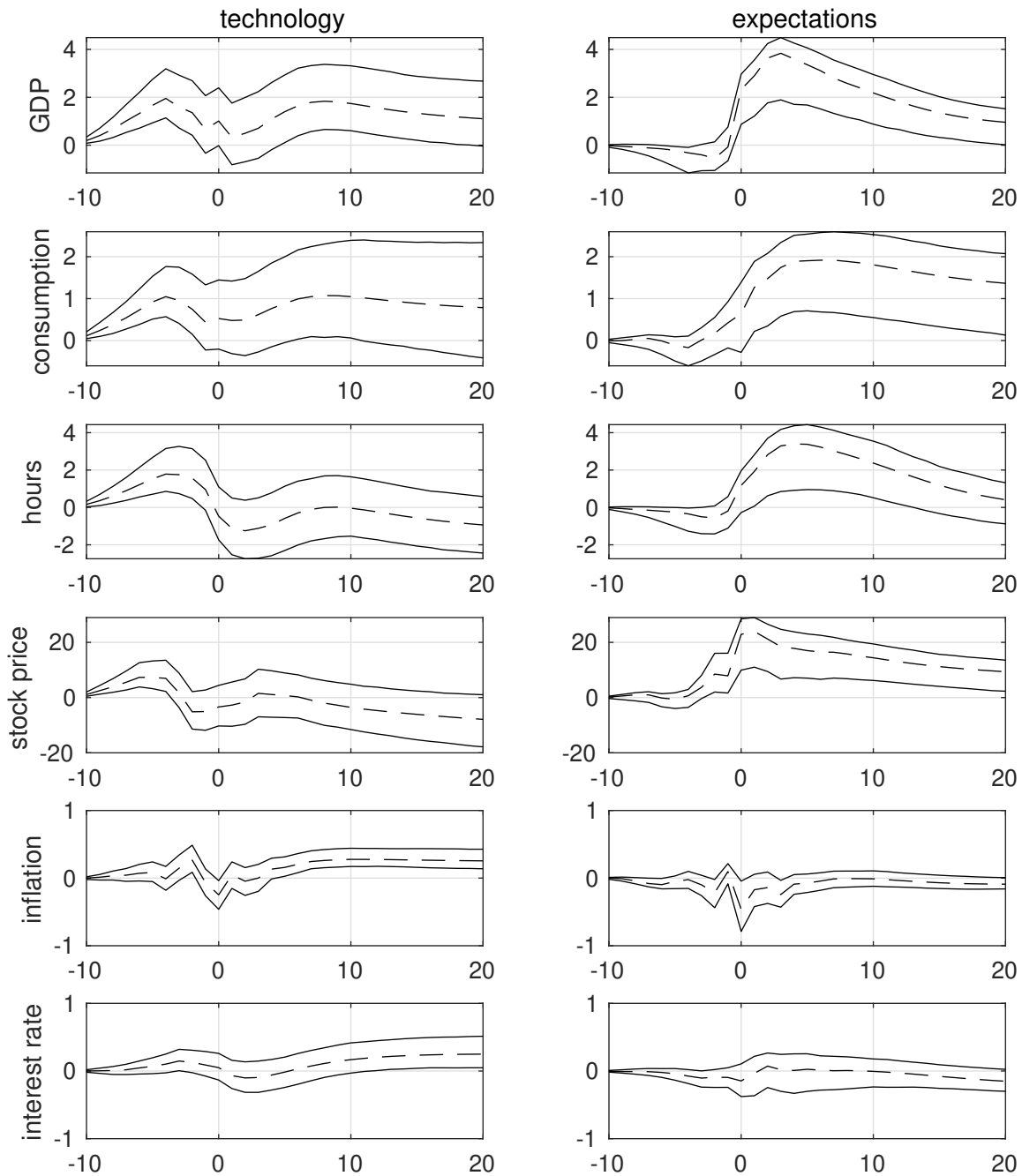


Figure 3: Impulse responses estimated from U.S. data. These figures display how the levels of six macroeconomic variables respond to a one standard deviation impulse in the technological disturbance (left column) and the expectational disturbance (right column) at time $t = 0$. The dashed lines are the point estimates and the solid lines represent the 90% bias-corrected bootstrap confidence interval. All units are annualized log points.

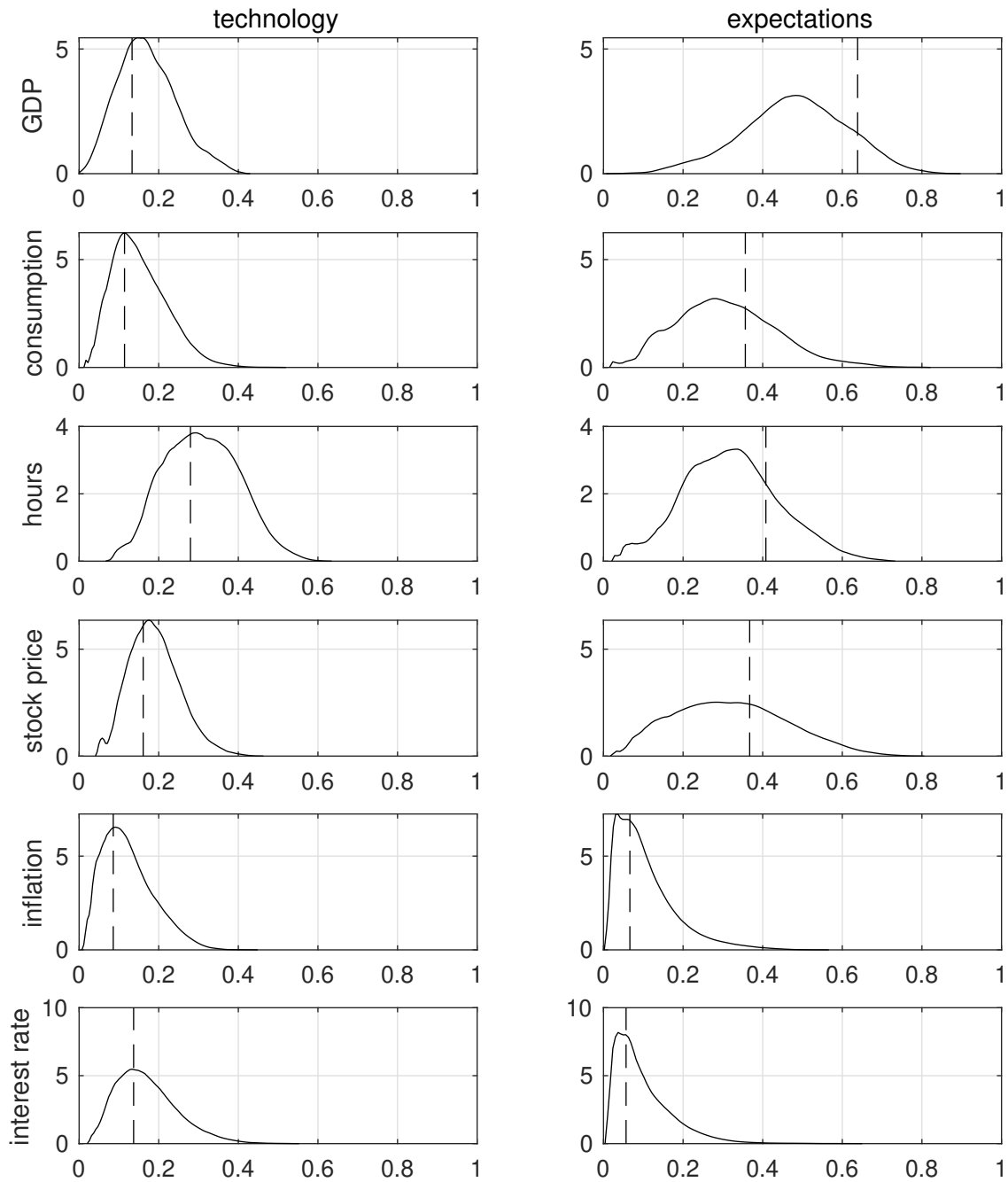


Figure 4: Variance shares estimated from U.S. data. These figures display the share of the variance in six macroeconomic variables attributable to disturbances in technology and expectations over business cycle frequencies (6-32 quarters). The dashed line is the point estimate and the solid line is the distribution of bias-corrected bootstrap estimates.

of these disturbances as noise in agents' signals of future technology. The exception to this pattern is in the stock price, which is marginally significant just prior to time zero, perhaps suggesting some deviations from rationality in stock price behavior.

Figure (4) plots the share of the variance in the six macroeconomic variables that is attributable to disturbances in technology and expectations over business cycle frequencies (6-32 quarters).¹⁶ The left column shows that technological disturbances themselves explain a relatively small portion of the business cycle fluctuations in all variables, exceeding 20% only in the case of hours. By contrast, expectational disturbances explain a large portion of the business-cycle variation in real variables: over 60% for GDP, over 40% for hours, and nearly 40% for consumption. These results are qualitatively similar to the findings of Blanchard et al. (2013) and Chahrour and Jurado (2018), both of which estimate a fully-structural equilibrium model with noise in agents' signal of future technology. Our results here contrast with these earlier papers, however, in finding larger effects of expectational disturbances for GDP relative to consumption. The remaining panels show that expectational disturbances explain only a modest portion of the business-cycle variation in stock prices, and even less for inflation and interest rates.

Overall, our results indicate that disturbances to expectations about future technology can play a relatively large role in explaining business cycle fluctuations in real variables. To a large degree, our findings are consistent with structural estimates of the importance of these disturbances in New Keynesian models. The small effect of expectational disturbances on inflation and interest rates is also consistent with the reason that these disturbances drive large fluctuations in such models; the propagation of disturbances to signal noise requires a high degree of nominal rigidity in prices and wages, and a weak response of interest rates. However, the fact that these disturbances are especially important for GDP does contrast with the findings of fully structural analyses, suggesting that the additional structure imposed by these models is not fully consistent with the observed data.

¹⁶The variance share is defined in (9). We compute this in the frequency domain by numerically integrating over the set $\Delta \equiv [-2\pi/6, -2\pi/32] \cup [2\pi/32, 2\pi/6]$.

7 Conclusion

Much of the empirical literature in macroeconomics is focused on recovering structural disturbances. In this paper, we have provided a formal condition which allows researchers to test whether the variables they have at hand are sufficient for this task. This condition holds in many cases where researchers have previously concluded that VAR approaches to recovering shocks were not feasible. We have suggested that allowing for non-invertible and non-causal disturbances can open the door to more plausible and more robust identification restrictions, especially those pertaining to expectations. While our empirical example focuses on disturbances to technology and expected technology, it would be natural to extend our analysis to other types of disturbances as well.

8 Data Availability

The data underlying this article are available in Zenodo at

<https://doi.org/10.5281/zenodo.4443539>.

References

- Alessi, L., M. Barigozzi, and M. Capasso (2011). Non-Fundamentalness in Structural Econometric Models: A Review. *International Statistical Review* 79(1), 16–47.
- Barsky, R. B., S. Basu, and K. Lee (2015). Whither News Shocks? In *NBER Macroeconomics Annual*, Volume 29, pp. 225–264. Chicago: University of Chicago Press.
- Barsky, R. B. and E. R. Sims (2011). News shocks and business cycles. *Journal of Monetary Economics* 58(3), 273–289.
- Barsky, R. B. and E. R. Sims (2012). Information, Animal Spirits, and the Meaning of Innovations in Consumer Confidence. *American Economic Review* 102(4), 1343–1377.
- Basu, S., J. G. Fernald, and M. S. Kimball (2006). Are Technology Improvements Contractionary? *American Economic Review* 96(5), 1418–1448.

- Beaudry, P., P. Fève, A. Guay, and F. Portier (2019). When is nonfundamentalness in SVARs a real problem? *Review of Economic Dynamics* 34, 221–243.
- Beaudry, P. and F. Portier (2006). Stock prices, news, and economic fluctuations. *American Economic Review* 96(4), 1293–1307.
- Bernanke, B. S. (1986). Alternative explanations of the money-income correlation. *Carnegie-Rochester Conference Series on Public Policy* 25, 49–99.
- Blanchard, O. and M. Watson (1986). Are Business Cycles All Alike? In *The American Business Cycle: Continuity and Change*, pp. 123–180. National Bureau of Economic Research, Inc.
- Blanchard, O. J., J. L’Huillier, and G. Lorenzoni (2013). News, Noise, and Fluctuations: An Empirical Exploration. *American Economic Review* 103(7), 3045–3070.
- Brockwell, P. J. and R. A. Davis (1991). *Time Series: Theory and Methods*. New York: Springer.
- Chahrour, R. and K. Jurado (2018). News or Noise? The Missing Link. *American Economic Review* 108(7), 1702–36.
- Fernández-Villaverde, J., J. F. Rubio-Ramírez, T. J. Sargent, and M. W. Watson (2007). ABCs (and Ds) of Understanding VARs. *American Economic Review* 97(3), 1021–1026.
- Forni, M. and L. Gambetti (2014). Sufficient information in structural vars. *Journal of Monetary Economics* 66, 124–136.
- Forni, M., L. Gambetti, M. Lippi, and L. Sala (2017a). Noise Bubbles. *The Economic Journal* 127(604), 1940–1976.
- Forni, M., L. Gambetti, M. Lippi, and L. Sala (2017b). Noisy News in Business Cycles. *American Economic Journal: Macroeconomics* 9(4), 122–52.
- Futia, C. A. (1981). Rational Expectations in Stationary Linear Models. *Econometrica* 49(1), 171–192.

- Hansen, L. P. and T. J. Sargent (1980). Formulating and estimating dynamic linear rational expectations models. *Journal of Economic Dynamics and Control* 2(1), 7–46.
- Hansen, L. P. and T. J. Sargent (1991). Two Difficulties in Interpreting Vector Autoregressions. In L. P. Hansen and T. J. Sargent (Eds.), *Rational Expectations Econometrics*, pp. 77–119. Boulder, CO: Westview Press.
- Jordà, Ò. (2005). Estimation and inference of impulse responses by local projections. *American Economic Review* 95(1), 161–182.
- Kilian, L. and H. Lütkepohl (2017). *Structural Vector Autoregressive Analysis*. Themes in Modern Econometrics. Cambridge University Press.
- Lanne, M. and P. Saikkonen (2013). Noncausal Vector Autoregression. *Econometric Theory* 29(3), 447–481.
- Lippi, M. and L. Reichlin (1993). The Dynamic Effects of Aggregate Demand and Supply Disturbances: Comment. *The American Economic Review* 83(3), 644–652.
- Lippi, M. and L. Reichlin (1994). VAR analysis, nonfundamental representations, blaschke matrices. *Journal of Econometrics* 63(1), 307–325.
- Mertens, K. and M. O. Ravn (2010). Measuring the Impact of Fiscal Policy in the Face of Anticipation: A Structural VAR Approach. *The Economic Journal* 120(544), 393–413.
- Nakamura, E. and J. Steinsson (2018). Identification in macroeconomics. *The Journal of Economic Perspectives* 32(3), 59–86.
- Plagborg-Møller, M. and C. K. Wolf (2018). Instrumental Variable Identification of Dynamic Variance Decompositions. Working paper, Princeton University.
- Quah, D. (1990). Permanent and Transitory Movements in Labor Income: An Explanation for “Excess Smoothness” in Consumption. *Journal of Political Economy* 98(3), 449–475.
- Rozanov, Y. A. (1967). *Stationary Random Processes*. San Francisco, CA: Holden-Day.

Schmitt-Grohé, S. and M. Uribe (2012). What’s News in Business Cycles. *Econometrica* 80(6), 2733–2764.

Sims, C. A. (1980). Macroeconomics and Reality. *Econometrica* 48(1), 1–48.

Stock, J. H. and M. W. Watson (2018). Identification and Estimation of Dynamic Causal Effects in Macroeconomics Using External Instruments. Working paper, Harvard University.

Uhlig, H. (2017). *Shocks, Sign Restrictions, and Identification*, Volume 2 of *Econometric Society Monographs*, pp. 95–127. Cambridge University Press.

A Proofs

We prove the results in this paper by first specifying the appropriate space of complex functions on which we will operate. The notation and terminology closely follows Rozanov (1967). We write the spectral representation of an arbitrary covariance stationary process $\{\xi_t\}$ as

$$\xi_t = \int_{-\pi}^{\pi} e^{i\lambda t} \Phi_{\xi}(d\lambda), \quad (10)$$

where $\Phi_{\xi}(d\lambda)$ is its associated random spectral measure. We say that a $1 \times n_{\xi}$ dimensional vector function $\psi(\lambda)$ belongs to the space $\mathcal{L}^2(F_{\xi})$ if

$$\int_{-\pi}^{\pi} \psi(\lambda) F_{\xi}(d\lambda) \psi(\lambda)^* \equiv \int_{-\pi}^{\pi} \sum_{k,l=1}^{n_{\xi}} \psi_k(\lambda) \overline{\psi_l(\lambda)} f_{\xi,kl}^{(\mu)}(\lambda) \mu(d\lambda) < \infty,$$

where $F_{\xi}(d\lambda) \equiv E\Phi_{\xi}(d\lambda)\Phi(d\lambda)^*$ denotes the spectral measure of $\{\xi_t\}$, $\mu(d\lambda)$ denotes any non-negative measure with respect to which all the elements of $F_{\xi}(d\lambda)$ are absolutely continuous, $f_{\xi,kl}^{(\mu)}(\lambda) = F_{\xi,kl}(d\lambda)/\mu(d\lambda)$ for $k, l = 1, \dots, n_{\xi}$, and the asterisk denotes complex conjugate transposition. If we define the scalar product

$$(\psi_1, \psi_2) = \int_{-\pi}^{\pi} \psi_1(\lambda) F_{\xi}(d\lambda) \psi_2(\lambda)^*,$$

and do not distinguish between two vector functions that satisfy $\|\psi_1 - \psi_2\| = 0$, then $\mathcal{L}^2(F_{\xi})$ becomes a Hilbert space.¹⁷

¹⁷The generalization of the Riesz-Fischer Theorem that is required to establish this fact is proven in Lemma 7.1, Ch. 1, of Rozanov (1967).

This space of functions is helpful for describing linear transformations between stationary processes. We say that $\{\eta_t\}$ can be obtained from $\{\xi_t\}$ by a *linear transformation* whenever it can be represented in the form

$$\eta_t = \int_{-\pi}^{\pi} e^{i\lambda t} \psi(\lambda) \Phi_{\xi}(d\lambda) \quad (11)$$

for all t , where $\psi(\lambda)$ is an $n_{\eta} \times n_{\xi}$ function with rows in $\mathcal{L}^2(F_{\xi})$. We call the function $\psi(\lambda)$ in this expression the *spectral characteristic* associated with the transformation. The following lemma says that recoverability is equivalent to the existence of a linear transformation of this type.

Lemma 1. *$\{\eta_t\}$ is recoverable from $\{\xi_t\}$ if and only if $\{\eta_t\}$ can be obtained from $\{\xi_t\}$ by a linear transformation of the type in equation (11).*

Proof of Lemma (1). First, we observe that $\mathcal{H}(\xi)$ is isomorphic to $\mathcal{L}^2(F_{\xi})$.¹⁸ This can be seen by defining a correspondence between elements $h \in \mathcal{H}(\xi)$ of the form

$$h = \int_{-\pi}^{\pi} \psi(\lambda) \Phi_{\xi}(d\lambda), \quad (12)$$

where

$$\int_{-\pi}^{\pi} |\psi_k(\lambda)|^2 F_{\xi, kk}(d\lambda) < \infty, \quad k = 1, \dots, n_{\xi}, \quad (13)$$

and the vector functions $\psi(\lambda) \in \mathcal{L}^2(F_{\xi})$ which occur in the representation (12). This correspondence is linear, since $h_1 \leftrightarrow \psi_1$ and $h_2 \leftrightarrow \psi_2$ implies

$$\alpha_1 h_1 + \alpha_2 h_2 = \int_{-\pi}^{\pi} (\alpha_1 \psi_1(\lambda) + \alpha_2 \psi_2(\lambda)) \Phi_{\xi}(d\lambda) \leftrightarrow \alpha_1 \psi_1 + \alpha_2 \psi_2$$

for arbitrary scalars α_1, α_2 . Moreover, it is isometric, because

$$(h_1, h_2) = \int_{-\pi}^{\pi} \psi_1(\lambda) F_{\xi}(d\lambda) \psi_2(\lambda)^* = (\psi_1, \psi_2).$$

Because the closed linear manifold spanned by elements of the form (12) coincides with $\mathcal{H}(\xi)$, and the closed linear manifold spanned by elements $\psi(\lambda)$ of the form (13) coincides with $\mathcal{L}^2(F_{\xi})$, it follows that correspondence we have defined can be extended by continuity to $\mathcal{H}(\xi)$ and $\mathcal{L}^2(F_{\xi})$, preserving both its linearity and isometry. Now we proceed to the main part of the proof.

¹⁸Recall that two Hilbert spaces are said to be “isomorphic” if it is possible to define a one-to-one correspondence between their elements which is linear and isometric.

Necessity: If $\mathcal{H}(\eta) \subseteq \mathcal{H}(\xi)$, then $\eta_{k,0} \in \mathcal{H}(\xi)$ for all $k = 1, \dots, n_\eta$. Because $\mathcal{H}(\xi)$ is isomorphic to $\mathcal{L}^2(F_\xi)$, there exists a unique vector function $\psi(\lambda)$, whose rows are elements of $\mathcal{L}^2(F_\xi)$, such that

$$\eta_0 = \int_{-\pi}^{\pi} \psi(\lambda) \Phi_\xi(d\lambda).$$

For every stationary process $\{\eta_t\}$, there exists a family of unitary operators U_t , $-\infty < t < \infty$, on $\mathcal{H}(\xi)$ such that

$$U_t \eta_{k,s} = \eta_{k,t+s}, \quad k = 1, \dots, n_\eta$$

for any t, s . To the unitary operator U_t in $\mathcal{H}(\eta)$ corresponds the operator of multiplication by $e^{i\lambda t}$ in $\mathcal{L}^2(F_\eta)$; that is, for all $k = 1, \dots, n_\eta$,

$$U_t \eta_{k,0} = U_t \left[\int_{-\pi}^{\pi} \delta_k \psi(\lambda) \Phi_\xi(d\lambda) \right] = \eta_{k,t} = \int_{-\pi}^{\pi} e^{i\lambda t} \delta_k \psi(\lambda) \Phi_\xi(d\lambda),$$

where δ_k is a $1 \times n_\eta$ constant vector with components $\delta_{kk} = 1$ and $\delta_{kl} = 0$ for $k \neq l$. From this it follows that η_t has a representation of the form (11).

Sufficiency: Suppose there exists a function $\psi(\lambda)$ with rows in $\mathcal{L}^2(F_\xi)$ such that equation (11) holds. Then the function $e^{i\lambda t} \delta_k \psi(\lambda)$ is evidently also an element of $\mathcal{L}^2(F_\xi)$ for each $k = 1, \dots, n_\eta$, since

$$\int_{-\pi}^{\pi} e^{i\lambda t} \delta_k \psi(\lambda) F_\xi(d\lambda) \psi(\lambda)^* \delta_k^* e^{-i\lambda t} = \int_{-\pi}^{\pi} \delta_k \psi(\lambda) F_\xi(d\lambda) \psi(\lambda)^* \delta_k^* < \infty.$$

Because $\mathcal{L}^2(F_\xi)$ is isomorphic to $\mathcal{H}(\xi)$, this means that $\eta_{k,t} \in \mathcal{H}(\xi)$ for $k = 1, \dots, n_\eta$. Therefore, $\mathcal{H}(\eta) \subseteq \mathcal{H}(\xi)$. \square

Our strategy for proving Theorem (1) is to project the values of the disturbance $\{\varepsilon_{k,t}\}$ at each point in time onto the space $\mathcal{H}(y)$ and then state necessary and sufficient conditions under which the projection error is zero. The following lemma states the optimal projection formula.

Lemma 2 (Optimal Smoothing). *The stationary process $\{\tilde{\varepsilon}_t\}$ consisting of the best linear estimates of $\{\varepsilon_t\}$ on the basis of the values $y_{k,s}$, $k = 1, \dots, n_y$, $-\infty < s < \infty$, is obtained from $\{y_t\}$ by a linear transformation of the form*

$$\tilde{\varepsilon}_t = \int_{-\pi}^{\pi} e^{i\lambda t} \varphi(\lambda)^\dagger \Phi_y(d\lambda).$$

Proof of Lemma (2). By Lemma (1), the projections $\tilde{\varepsilon}_{k,t}$ form an n_ε dimensional stationary process $\{\tilde{\varepsilon}_t\}$ which is obtained from the process $\{y_t\}$ by a linear transformation,

$$\tilde{\varepsilon}_t = \int_{-\pi}^{\pi} e^{i\lambda t} \psi(\lambda) \Phi_y(d\lambda),$$

where $\psi(\lambda)$ is some $n_\varepsilon \times n_y$ function whose rows are elements of $\mathcal{L}^2(F_y)$. For the prediction errors $\varepsilon_{k,t} - \tilde{\varepsilon}_{k,t}$, $k = 1, \dots, n_\varepsilon$, to be orthogonal to the space $\mathcal{H}(y)$, it must be that

$$E[(\varepsilon_t - \tilde{\varepsilon}_t)y_s^*] = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda(t-s)} [\varphi(\lambda)^* - \psi(\lambda)\varphi(\lambda)\varphi(\lambda)^*] d\lambda = 0$$

for any t and s . This is true if and only if

$$\varphi(\lambda)^* = \psi(\lambda)\varphi(\lambda)\varphi(\lambda)^* \quad (14)$$

for almost all λ . By definition, $\psi(\lambda) = \varphi(\lambda)^\dagger$ is a solution. Moreover, this solution is unique, in the sense that its rows are uniquely determined as elements of the space $\mathcal{L}^2(F_y)$. To see this, consider any other function, $\psi(\lambda) \neq \varphi(\lambda)^\dagger$, whose rows are elements of $\mathcal{L}^2(F_y)$, which also satisfies (14). Then

$$\|\delta_k \varphi(\lambda)^\dagger - \delta_k \psi(\lambda)\|^2 = \int_{-\pi}^{\pi} \delta_k (\varphi(\lambda)^\dagger - \psi(\lambda)) \varphi(\lambda) \varphi(\lambda)^* (\varphi(\lambda)^\dagger - \psi(\lambda))^* \delta_k^* d\lambda = 0$$

for each $k = 1, \dots, n_\varepsilon$, where δ_k denotes a $1 \times n_\varepsilon$ constant vector with components $\delta_{kk} = 1$ and $\delta_{kl} = 0$ for $k \neq l$. \square

Proof of Theorem (1). Using the optimal smoothing formula from Lemma (2),

$$\|\varepsilon_{k,t} - \tilde{\varepsilon}_{k,t}\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta_k (I_{n_\varepsilon} - \varphi(\lambda)^\dagger \varphi(\lambda)) (I_{n_\varepsilon} - \varphi(\lambda)^\dagger \varphi(\lambda))^* \delta_k^* d\lambda,$$

which equals zero if and only if $\delta_k (I_{n_\varepsilon} - \varphi(\lambda)^\dagger \varphi(\lambda)) = 0$ almost everywhere. \square

Our strategy for proving Theorem (3) is to project the values of the disturbance $\{\varepsilon_{k,t}\}$ at each point in time onto the space $\mathcal{H}_t(y)$ and then state necessary and sufficient conditions under which the projection error is zero. We first prove a lemma which states the optimal projection formula. To do so, we introduce some additional notation: for any function $\varphi(\lambda)$ with Fourier expansion

$$\varphi(\lambda) = \sum_{s=-\infty}^{\infty} \varphi_s e^{-i\lambda s},$$

let $[\varphi(\lambda)]_+$ denote the function obtained by removing all the Fourier coefficients associated with negative values of s ,

$$[\varphi(\lambda)]_+ = \sum_{s=0}^{\infty} \varphi_s e^{-i\lambda s}.$$

Lemma 3 (Optimal Filtering). *The stationary process $\{\hat{\varepsilon}_t\}$ consisting of the best linear estimates of $\{\varepsilon_t\}$ on the basis of the values $y_{k,s}$, $k = 1, \dots, n_y$, $-\infty < s \leq t$, is obtained from $\{y_t\}$ by a linear transformation of the form*

$$\hat{\varepsilon}_t = \int_{-\pi}^{\pi} e^{i\lambda t} [\varphi(\lambda)^\dagger \gamma(\lambda)]_+ \gamma(\lambda)^\dagger \Phi_y(d\lambda),$$

where $\gamma(\lambda)$ comes from some version of Wold's decomposition of $\{y_t\}$.

Proof of Lemma (3). First we observe that the projections of $\varepsilon_{k,t}$ and $\tilde{\varepsilon}_{k,t}$ on $\mathcal{H}_t(y)$ coincide. Combining the representation of $\{\tilde{\varepsilon}_t\}$ from Lemma (2) with the Wold representation of $\{y_t\}$ in equation (3), we obtain

$$\tilde{\varepsilon}_t = \int_{-\pi}^{\pi} e^{i\lambda t} \varphi(\lambda)^\dagger \gamma(\lambda) \Phi_w(d\lambda).$$

Using this representation of $\{\tilde{\varepsilon}_t\}$, we can see that the projections $\hat{\varepsilon}_{k,t}$ form a stationary process $\{\hat{\varepsilon}_t\}$ which is obtained from $\{w_t\}$ by a linear transformation of the form

$$\hat{\varepsilon}_t = \int_{-\pi}^{\pi} e^{i\lambda t} [\varphi(\lambda)^\dagger \gamma(\lambda)]_+ \Phi_w(d\lambda).$$

Since $\gamma(\lambda)$ has full column rank for almost all λ , it follows that $\gamma(\lambda)^\dagger \gamma(\lambda) = I_{r_y}$, where r_y is the rank of $f_y(\lambda)$. Therefore

$$\Phi_w(d\lambda) = \gamma(\lambda)^\dagger \Phi_y(d\lambda).$$

Substituting this into the previous expression for $\Phi_w(d\lambda)$ gives the linear transformation reported in the lemma. Analogously to the proof of Lemma (2), the uniqueness of the projections $\hat{\varepsilon}_{k,t}$ implies that the spectral characteristic in this representation has rows which are all unique elements of $\mathcal{L}^2(F_y)$. \square

Proof of Theorem (3). By Theorem (1), recoverability of $\{\varepsilon_{k,t}\}$ means that

$$\varepsilon_{k,t} = \int_{-\pi}^{\pi} e^{i\lambda t} \delta_k \varphi(\lambda)^\dagger \Phi_y(d\lambda) = \int_{-\pi}^{\pi} e^{i\lambda t} \delta_k \varphi(\lambda)^\dagger \gamma(\lambda)^\dagger \Phi_w(d\lambda),$$

where the second equality uses Wold's decomposition of $\{y_t\}$. Combining this with the optimal filtering formula from Lemma (3),

$$\|\varepsilon_{k,t} - \hat{\varepsilon}_{k,t}\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta_k(\varphi(\lambda)^\dagger \gamma(\lambda) - [\varphi(\lambda)^\dagger \gamma(\lambda)]_+) (\varphi(\lambda)^\dagger \gamma(\lambda) - [\varphi(\lambda)^\dagger \gamma(\lambda)]_+)^* \delta_k^* d\lambda,$$

which equals zero if and only if $\delta_k[\varphi(\lambda)^\dagger \gamma(\lambda)]_+ = \delta_k \varphi(\lambda)^\dagger \gamma(\lambda)$ for almost all λ . In other words, the Fourier coefficients of $\delta_k \varphi(\lambda)^\dagger \gamma(\lambda)$ must vanish for all $s < 0$, which is the condition stated in the theorem. \square

B Monte Carlo

To evaluate the ability of our procedure to uncover technological and expectational disturbances in practice, we perform two Monte Carlo exercises. In the first, we apply our procedure to model-generated data with a large sample (100,000 time periods) to show that, in population, the procedure nearly perfectly recovers the true structural impulse responses. In the second, we apply the procedure to one thousand artificial samples of the same length as our actual data ($T = 284$), and find our approach does a good job of recovering the truth on average.

As a data generating process, we use the equilibrium model estimated in section (E) of Chahrour and Jurado (2018). This model includes a standard set of frictions, such as nominal wage and price rigidities, external consumption habits, and investment adjustment costs. The model includes disturbances to technology, expectations of future technology, government spending, and monetary policy. Exogenous policy processes are AR(1), while the process for technology and associated information is a special case of the information structure in section (3) of this paper, with

$$\alpha_0 = \sigma_a \quad \text{and} \quad \alpha_k = 0 \text{ for all } k \neq 0,$$

$$\varsigma_k = \sigma_a \left(\frac{1 - \rho}{1 + \rho} \right) \rho^k \text{ for all } k \geq 0$$

$$\nu_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda k} \sigma_v (1 - e^{-i\lambda}) \frac{1 - 2(\beta + \bar{\beta})e^{-i\lambda} + |\beta|^2 e^{-i\lambda^2}}{1 - 2\rho e^{-i\lambda} + \rho^2 e^{-i\lambda^2}} d\lambda.$$

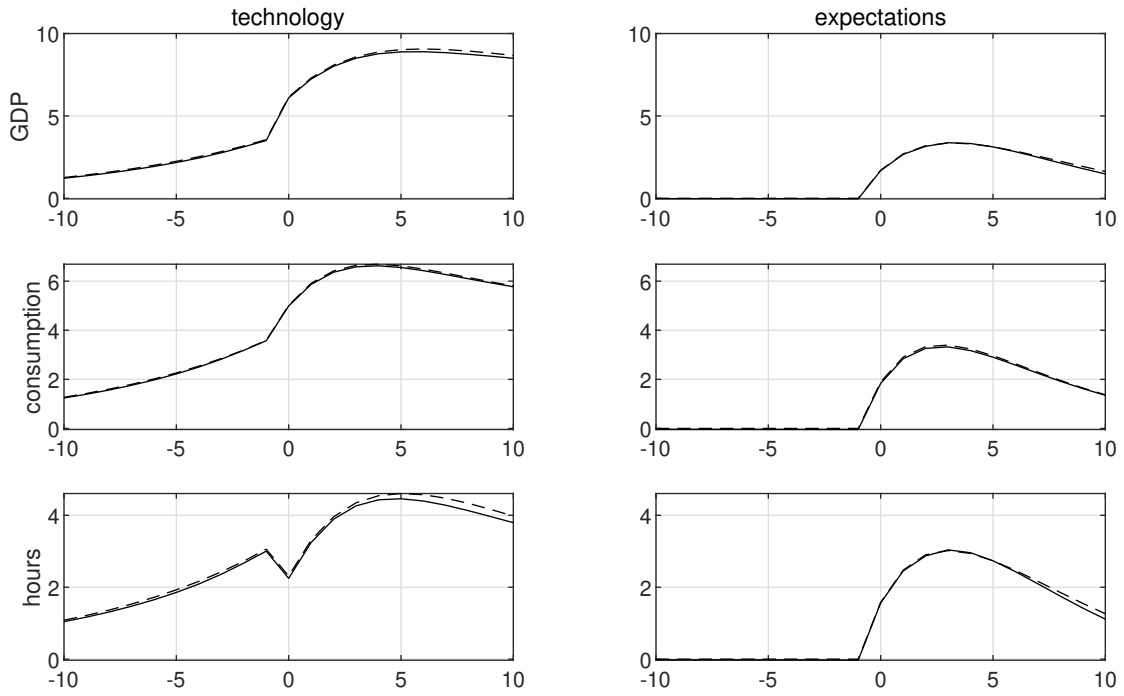
There is also a nonlinear restriction on the parameters σ_a , σ_v , ρ , and β , which ensures that $\{a_t\}$ can be written as the sum of a permanent component with first-order autoregressive dynamics in first differences, and a transitory component with first-order

autoregressive dynamics in levels. Such information structures appear frequently in the related literature. For parameter values, we use the maximum likelihood estimates in column 2 of table 6 in the online appendix of Chahrour and Jurado (2018).

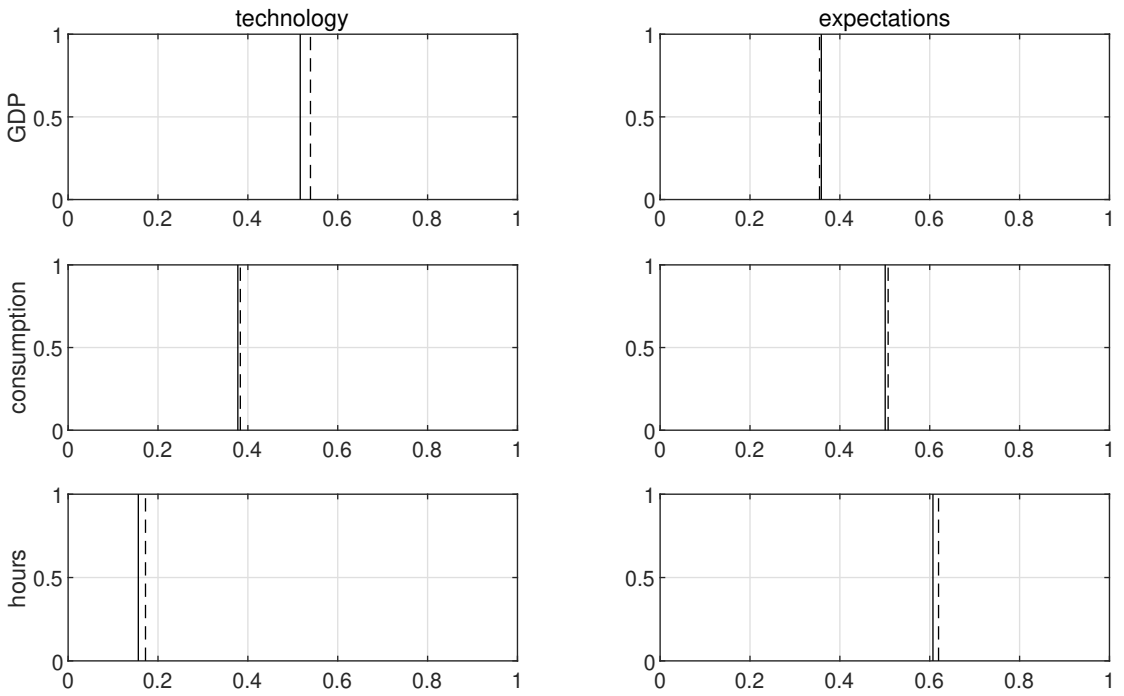
From the model, we simulate data on four variables: GDP, consumption, hours, and technology. The VAR procedure we apply is the one that is described in the main text. Before applying the procedure, we check recoverability of the four structural disturbances. To do this, we use the equilibrium state-space representation to construct the function $\varphi(\lambda)$ from Theorem (1). We then draw a random value $\lambda_u = 0.4431$, and find that null space of $\varphi(\lambda_u)$ is empty ($|\varphi(\lambda_u)| = -0.2186 - 0.0827i$). Hence, by Theorem (1) we conclude that all four structural disturbances are recoverable.

Panels (a) and (b) of Figure (5) compare the point estimates from the long sample and the true, model-implied responses. The two are nearly identical in all cases, indicating that — even though our model does not have an exact VAR representation — a fourth-order VAR almost perfectly approximates the time series properties of the generated data. We conclude that, in population, our approach is well-suited to identifying the true impulse responses.

Panels (a) and (b) of Figure (6) compare the distribution of estimated impulse responses and variances contribution across the 1000 simulations of length $T = 284$, with the corresponding true, model-implied values. In virtually all cases, the truth lies within the simulated bounds. There is evidence of a downward finite-sample bias both in the impulse responses and the variance shares, which is somewhat more pronounced in the case of the expectational disturbance. Such a finite-sample bias often appears in the presence of persistent observable variables. This is why, in the results we report in the main text, we bias adjust our bootstrap estimates.

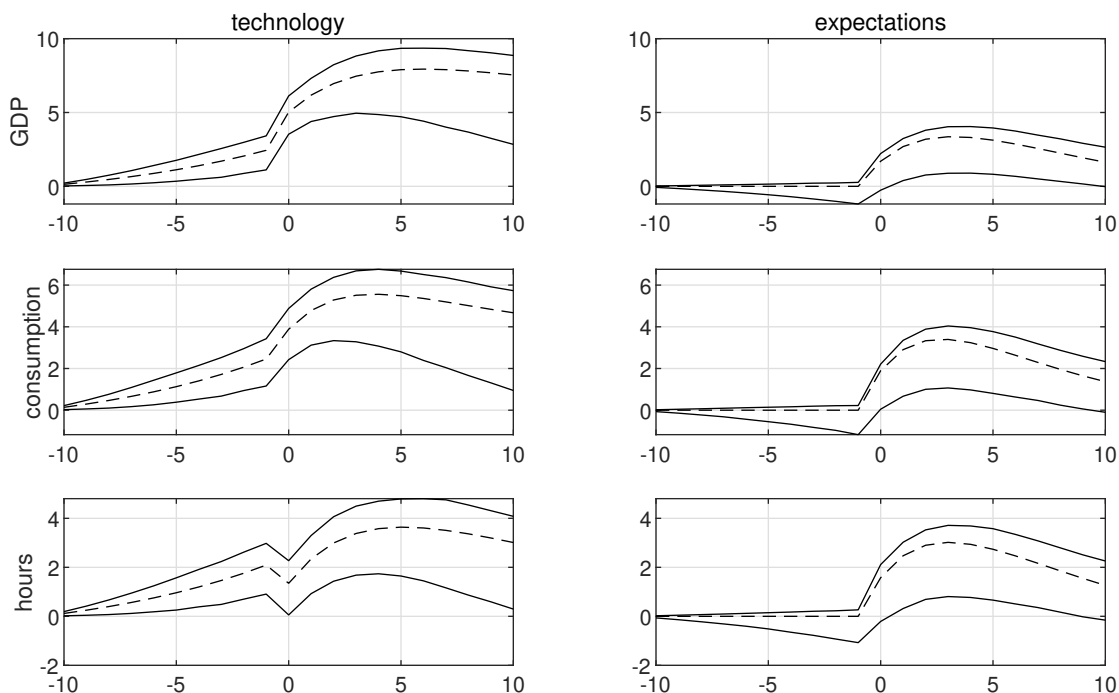


(a) Impulse responses

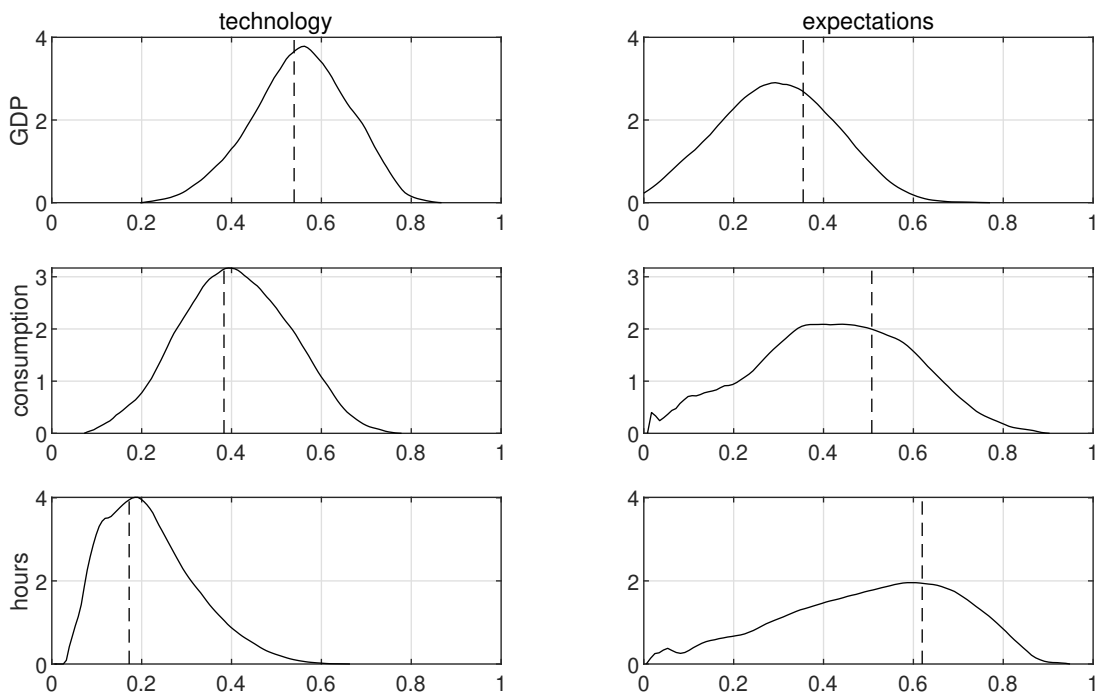


(b) Variance shares

Figure 5: Large sample Monte Carlo. Solid lines are the estimates and dashed lines are the truth.



(a) Impulse responses



(b) Variance shares

Figure 6: Small sample Monte Carlo. Solid lines in (a) are 90% bands across 1000 simulations and in (b) are distributions of the estimates. Dashed lines are the truth.